

Random k -SAT and the Power of Two Choices

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September 25, 2012

Abstract

We study a semi-random model for k -SAT involving an Achlioptas-process version of the random k -SAT process: a bounded number of k -CNF clauses are drawn uniformly at random at each step, and exactly one added to the growing formula according to a particular rule. We show the validity of the model by proving the existence of a rule that shifts the satisfiability threshold. This extends a well-studied area of probabilistic combinatorics (Achlioptas processes) to random CSP's. In particular, while a rule to delay the 2-SAT threshold was known previously, this is the first proof of a rule to shift the threshold of a CSP that is NP-hard.

We then propose a gap decision problem based upon this semi-random model. The aim of the problem is to investigate the hardness of the random k -SAT decision problem, as opposed to the problem of finding an assignment or certificate of unsatisfiability. While the usual decision problem is trivial in some respects because of the sharp threshold, in this semi-random model, with an adversary who can shift the threshold, the problem becomes relevant. Finally, we discuss connections to the study of Achlioptas random graphs.

1 Introduction

The mathematical study of phase transitions and threshold behavior in random structures began with Erdős and Rényi's first paper on random graphs [22]. In it, they showed that for any fixed $\epsilon > 0$, if the number of edges in the random graph is at most $(1/2 - \epsilon)n$ then the largest connected component is of size $O(\log n)$ with probability $1 - o(1)$ (whp), and if the number of edges is at least $(1/2 + \epsilon)n$, then the largest component is of size $\Theta(n)$ whp. The existence of a giant, linear-sized component thus exhibits what is known as a sharp threshold, with its probability rising from near 0 to near 1 with the addition of a sublinear number of additional edges. This is as opposed to the coarse threshold for the presence of a triangle in a random graph, a property whose probability is strictly bounded away from 0 and 1 for any linear number of edges. Since that initial paper, the random graph phase transition has been studied in great detail; it is now known that the scaling window has width $n^{2/3}$ [13], the structure of both the giant component and the smaller components is well-understood, and many modifications of the original model have been studied. The Erdős-Rényi random graph still plays a central role in discrete probability, both as an object in its own right and as a tool to solve other problems.

In theoretical computer science, the threshold phenomenon that has attracted the most study is the unsatisfiability threshold in the random k -SAT model. An instance of random k -SAT on n variables and m clauses consists of the conjunction of m clauses, each of which is the disjunction of k literals and chosen uniformly at random from the set of $\binom{n}{k}2^k$ possible clauses. Small variations in the description of the model, such as the difference between adding clauses with or without replacement are not significant with respect to the threshold behavior.

In [27] Friedgut proved that the satisfiability threshold is sharp. In particular, he proved that there exists a sequence $r_k(n)$ so that for every $\epsilon > 0$,

$$\begin{aligned}\Pr \left[\Phi_{(r_k(n)+\epsilon)n}^{(k)} \text{ is satisfiable} \right] &= o(1) \\ \Pr \left[\Phi_{(r_k(n)-\epsilon)n}^{(k)} \text{ is satisfiable} \right] &= 1 - o(1)\end{aligned}$$

where $\Phi_{rn}^{(k)}$ is a uniformly random k -SAT formula with rn clauses. It remains an open problem whether or not the sequence $r_k(n)$ has a limit; this is sometimes referred to as the ‘satisfiability threshold conjecture’. Much work has been done on proving upper and lower bounds on $r_k(n)$; the current best upper and lower bounds for $k = 3$ are 4.508 [21] and 3.52 [31], [28] respectively. See [3] for best current bounds for other values of k and a survey of the problem. Typically upper bounds come from a variant of the first-moment method, while lower bounds come from either analyzing an algorithm [16], [17], [1] or the second-moment method [4].

A second central open question in this area is whether or not random k -SAT formulae ($k \geq 3$) at or near the satisfiability threshold are computationally hard (2-SAT is solvable in polynomial time in the worst case). Selman, Mitchell and Levesque [37] gave experimental evidence that near the threshold it is in fact difficult to determine satisfiability. But in some sense, the decision version of the random k -SAT problem is trivially easy. Above the threshold density we can say ‘unsatisfiable’ and below the threshold say ‘satisfiable’ and be correct with high probability

without even looking at the sampled instance. Because of this, the study of the hardness of random k -SAT has turned towards finding certificates of unsatisfiability for clause densities above but as close as possible to the threshold [23], [25], [19].

In this work we consider a variant of the random k -SAT problem for which the decision problem is still relevant. Our initial motivation came from the work on Achlioptas processes in the study of random graphs. Dimitris Achlioptas initiated this study by asking whether, given the choice of two random edges at each step of a random graph process, one could delay the phase transition by a constant factor. Formally an Achlioptas random graph process is defined as follows:

- Begin at step 0 with an empty graph on n vertices.
- At step i , two uniformly random edges are presented, and exactly one of the two is selected according to a given rule and added to the graph
- The choice of edge can depend on the edges presented, the current graph, and the history of the process, but not on the edges to be presented in subsequent steps.

His original question was whether there is a rule for choosing one of the two edges so that at step $(1/2 + \epsilon)n$ the graph contains no linear-sized connected component whp. His question was motivated by the ‘power of two choices’ in load balancing [6], [35] and was answered affirmatively by Bohman and Frieze [9] in the first of many papers study Achlioptas processes. While the phase transition has received the most attention ([12], [26], [11]), rules for shifting the threshold of other properties have also been found (e.g. Hamiltonian cycles [33] or small subgraphs [32], [36]). One primary aim of the study of Achlioptas processes is to understand which qualitative properties of the phase transition are robust under small modifications of the model. In [29], [30], [7] it is shown that certain critical exponents are universal for a large class of Achlioptas rules.

Sinclair and Vilenchik [38] first considered the Achlioptas process model with regard to a random CSP. In particular, they exhibit a rule for choosing one of two random clauses that delays unsatisfiability for random 2-SAT by a constant factor. They also consider ‘off-line’ rules for random k -SAT and on-line rules for k -SAT with $k = \omega(\log n)$.

In this work we study the Achlioptas-process version of random k -SAT (see Section 2 for a formal definition), and show in Section 3 that in fact the satisfiability threshold can be shifted, for any k , and in particular for the computationally interesting cases $k \geq 3$. Studying this semi-random model of k -SAT is a step towards understanding the standard model better, as has been done in the case of random graphs, but we also argue that this model is particularly relevant for k -SAT because of the computational aspect.

We aim to address the question of the hardness of the random k -SAT decision problem. To avoid the triviality of almost sure satisfiability below the threshold and almost sure unsatisfiability above the threshold, we use the semi-random model of k -SAT described above and propose an accompanying decision problem in Section 4.

To sum up, this paper makes four main contributions:

1. We introduce a semi-random model of k -SAT and show that with respect to the satisfiability threshold the model is not trivial.
2. We propose a gap decision problem for this semi-random model that may be more approachable than random k -SAT from the perspective of computational complexity.
3. We prove that a specific Achlioptas rule can shift the satisfiability of random k -SAT for all $k \geq 2$. This was previously known only for the case of 2-SAT.
4. The rule we choose and our method of proof also shows that biased random k -SAT formulas are easier to satisfy than unbiased ones.

Notation

The set of binary variables on which our random formulae are built is $\{x_1, \dots, x_n\}$, and all asymptotics are as $n \rightarrow \infty$. A literal is a variable x_i or its negation \bar{x}_i , and we will denote literals with the letter w . A k -clause is a disjunction of k literals, $(w_{i_1} \vee w_{i_2} \vee \dots \vee w_{i_k})$. A formula of m clauses is the conjunction of m k -clauses and is satisfiable if there exists an assignment to the n variables that satisfies each of the m clauses. We will write Φ_m for a formula of m clauses. We write that an event E holds with high probability or whp if $\Pr[E] \rightarrow 1$ as $n \rightarrow \infty$.

2 Semi-Random k -SAT: The Model

Here we define an l -clause Achlioptas k -SAT process analogously to an Achlioptas random graph process:

1. Begin at step 0 with an empty formula, $\Phi_0 = \emptyset$.
2. Each step, l clauses are selected uniformly at random, with replacement, from the $\binom{n}{k} 2^k$ possible k -CNF clauses.
3. According to a fixed rule R , exactly one of the l clauses is chosen and added to the current formula. $\Phi_i = \Phi_{i-1} \wedge \phi_i$, where Φ_{i-1} is the current formula, and ϕ_i is the clause chosen at step i .

Note that different rules R lead to a different processes (and different distributions over formulas Φ_m at step m). The rule ‘Always select the first clause’ leads to the classic random k -SAT distribution. The rule R can be a function of the l clauses presented, the current formula Φ_{i-1} , and the entire history of presented clauses up to step i . The rule can also use randomness. The rule, however, cannot be a function of the clauses presented in subsequent steps (such rules, while not standard Achlioptas processes, are called ‘off-line’ rules and have been studied for both the random graphs [11] and k -SAT formulae [38]). The rules we analyze below will be

simpler: they will be functions of only the current formula and the l clauses presented in the current round.

Semi-random models of NP-hard CSP's have been proposed and studied before. Blum and Spencer [8] describe a semi-random model for graph coloring in which an adversary's edge choices are reversed at random with some probability, and they give algorithms for coloring in a range of the parameters. Feige and Krauthgamer [24] give an algorithm for finding planted cliques in a semi-random graph in which an adversary is allowed to remove arbitrary edges from among the non-planted edges. In [10] Bohman et al. consider smoothed graphs, with random edges added to arbitrary graphs, and they determine how many random edges must be added for various monotone graph properties to hold whp. Krivelevich, Sudakov and Tetali [34] consider Ramsey properties in this model and study an analogous smoothed model of k -SAT formulae.

3 Results

Our first result is that the satisfiability threshold for random k -SAT can be delayed with an l -clause rule, for constant l .

Theorem 1. *For every integer $k \geq 2$, there exists an integer l and an l -clause Achlioptas rule for random k -SAT so that with probability $1 - o(1)$, the formula generated after $2^{k+1} \ln 2 \cdot n$ steps is satisfiable.*

In particular, since the threshold for random k -SAT is at most $2^k \ln 2$, we show that with bounded choice, unsatisfiability can be delayed by a constant factor.

Next we specialize to the case $k = 3$.

Theorem 2. *There exists a 5-clause Achlioptas rule for random 3-SAT so that with probability $1 - o(1)$, the formula generated after $5.065n$ steps is satisfiable.*

In particular, 5.065 is above the best upper bound for the random 3-SAT threshold, and so we have in fact shifted the threshold.

For the case $k = 2$, we improve the constant factor of delay for a 2-clause rule in the results of [38] with a different rule and a different proof.

Theorem 3. *There is a 2-clause Achlioptas rule for random 2-SAT that generates a formula that, after $1.055n$ steps, is satisfiable whp.*

The proofs will follow in Section 5.

4 Semi-Random Gap k -SAT

Once we know that an Achlioptas rule can change the satisfiability threshold, the following decision problem becomes meaningful.

Let $\Phi_i, i = 1, \dots$ be a growing semi-random 3-SAT formula generated according to a 2-clause Achlioptas rule R . We want to distinguish between the following two cases:

- NO: At step $4n$, the formula is unsatisfiable.
- YES: At step $5n$, the formula is satisfiable.

If neither condition holds, then either answer is accepted.

Question 1. *Is there an efficient algorithm that for all rules R gives an acceptable answer to the above problem with probability $1 - o(1)$?*

We can generalize the above problem to k -SAT and by varying three parameters: the number of random clauses presented at each step (l), the lower threshold (c_1), and the upper threshold (c_2). We have chosen $k = 3, l = 2, c_1 = 4, c_2 = 5$ for simplicity. The problem becomes harder as l increases, and easier as either c_2 increases or c_1 decreases. If the adversary had no choice, and a random clause was added at each step, the problem would be easy: the satisfiability threshold would occur at $r_k(n)$, and if $r_k(n) \leq c_1$, NO would be acceptable whp; if $r_k(n) \geq c_2$ YES would be acceptable; and otherwise either answer would be acceptable. But because we show that the adversary can in fact shift the threshold, the problem is no longer trivial.

One way to interpret Achlioptas' original question is whether, under a specific model of semi-random graphs, the phase transition occurs when the average degree of a vertex hits 1 as it does in the Erdős-Rényi random graph. Bohman and Frieze answered 'no' to this question, and in some sense this shows that average degree 1 in the standard model is an artifact of the independence and uniformity of the random edges. However, the study of Achlioptas processes has identified a different statistic, rather than the average degree, that does control the phase transition for a large subclass of Achlioptas processes. This is the *susceptibility*, or the average component size in the graph: $S(G) = n^{-1} \sum_v |C(v)|$. Bohman and Kravitz [12] and Spencer and Wormald [39] show that for the class of 'bounded-size' rules, the blow-up point of an ODE tracking the growth of $S(G)$ marks the critical point for the phase transition.

The susceptibility allows one to understand where and why the phase transition occurs in Achlioptas random graph processes but is not needed algorithmically, since detecting a giant component is already an easy computational problem. In the case of k -SAT however, if there was such a statistic, correlated with the threshold, which was efficiently computable, then the decision problem version of random k -SAT would be tractable for a non-trivial reason.

5 Proofs

Theorem 1 is corollary of the following lemma:

Lemma 1. *For fixed integers $k \geq 2$ and $l \geq 2$, there exists an l -clause Achlioptas rule for random k -SAT which creates a formula that, for every $\epsilon > 0$, is satisfiable whp after $(r(k, l) - \epsilon)n$ steps,*

where

$$r(k, l) = \frac{2^{kl/2}}{4(k+1)^{(l-1)/2}} \quad (1)$$

In particular, for every k there is an l large enough so that there exists an l -clause rule that delays unsatisfiability.

We start by giving an idea of the strategy we will use to delay unsatisfiability. Consider a biased a random 3-SAT formula generated by adding $m = rn$ clauses, with each clause selected as follows: with probability p , choose a clause uniformly from all 3-clauses with 3 positive literals, and with probability $1 - p$ choose a clause uniformly from all 3-clauses. Let Φ_{rn} denote that random formula. This formula is biased in favor of assignments with more +1's than -1's. Let Z_β be the number of satisfying assignments to Φ_{rn} with βn +1's and $(1 - \beta)n$ -1's, and let x_β be the particular assignment that assigns the first βn variables +1 and the rest -1. Then

$$\begin{aligned} \frac{1}{n} \log \mathbb{E} Z_\beta &= \frac{1}{n} \log \left(\binom{n}{\beta n} \Pr[x_\beta \text{ satisfying}] \right) \\ &= H(\beta) + r \log \Pr[x_\beta \text{ satisfies } \Phi_1] + o(1) \end{aligned}$$

where $H(\beta)$ is the binary entropy function.

$$\Pr[x_\beta \text{ satisfies } \Phi_1] \sim p(1 - (1 - \beta)^3) + (1 - p) \cdot 7/8$$

If we pick r small enough so that $\max_{\beta \in (0,1)} H(\beta) + r \log[p(1 - (1 - \beta)^3) + (1 - p) \cdot 7/8] > 1$, then there will be exponentially many satisfying assignments in expectation. One can show that for any $p > 0$, there is some such r larger than $\log 2 / \log(7/8) \sim 5.19...$ which is the simple first-moment upper bound for random 3-SAT. And in fact for any $r > 0$, there is a $p \in (0, 1)$ so that $\max_{\beta \in (0,1)} H(\beta) + r \log[p(1 - (1 - \beta)^3) + (1 - p) \cdot 7/8] > 1$, which shows that with enough bias, the first-moment bound can be pushed arbitrarily high. This complements results on random regular k -SAT [15], in which an extreme lack of bias, with every literal having the same degree, leads to an earlier threshold.

Having exponentially many solutions in expectation does not imply a single solution with significant probability, so to prove the lemma we will use a related but different rule. This rule will select one of the l clauses presented at each step as follows:

- If one of the first $l - 1$ clauses contains at least two positive literals, add it. (If there is more than one such clause, add the first.)
- Otherwise add clause l .

The effect of this rule is to bias the formula in favor of majority +1 assignments as above. Let $r = r(k, l) - \epsilon$. To prove that this rule produces a satisfiable formula whp, we will begin by taking the formula at step rn and converting it into a 2-SAT formula. For each clause with two or more positive literals, we keep a 2-CNF clause with only the first two positive literals; for each clause with exactly one positive literal, we keep a 2-clause with the positive literal and the first

negative literal; and for each clause with all negative literals, we keep a 2-clause with the first two literals. This gives a 2-SAT formula with rn clauses, call it $F_2(rn)$. If $F_2(rn)$ is satisfiable, the original formula is also satisfiable since each 2-clause is a sub-clause of the corresponding k -clause. Each clause is also added independently, and the probability it has two, one or zero positive literals is, respectively:

$$\begin{aligned} p_2 &= 1 - \left(\frac{k+1}{2^k} \right)^l \\ p_1 &= \left(\frac{k+1}{2^k} \right)^l \cdot \frac{k}{k+1} \\ p_0 &= \left(\frac{k+1}{2^k} \right)^l \cdot \frac{1}{k+1} \end{aligned}$$

Also, each clause with i positive literals is distributed uniformly from all 2-clauses with i positive literals. As is standard in studying random graphs and random CSP's, we will consider the model of random 2-CNF formulas in which each of the $\binom{n}{2}$ possible clauses with two positive literals is present in a random formula $\overline{F}_2(r, n)$ independently with probability q_2 , each of the $n(n-1)$ clauses with one positive and one negative literal is present with probability q_1 , and each of the $\binom{n}{2}$ with two negative literals is present independently with probability q_0 . If we set

$$\begin{aligned} q_2 &= \frac{2p_2r}{n} \\ q_1 &= \frac{p_1r}{n} \\ q_0 &= \frac{2p_0r}{n} \end{aligned}$$

then proving satisfiability whp of $\overline{F}_2(r, n)$ implies satisfiability whp of $F_2(rn)$ (see [14], Appendix A, for details of the equivalent behavior of the two models).

To study the satisfiability of $\overline{F}_2(r, n)$ we will use an approach of [18] and [20], following [5], where it is noted that if there is no bicycle in a formula's 'implication graph', the formula is satisfiable. The vertices of the implication graph are the $2n$ literals, and for each clause $(w_i \vee w_j)$ in the formula, we add two directed edges, $(\overline{w}_i \rightarrow w_j)$ and $(\overline{w}_j \rightarrow w_i)$ to the graph. A bicycle of length k is a sequence of k literals of distinct variables, w_1, w_2, \dots, w_k where the $k-1$ directed edges $(w_1 \rightarrow w_2), (w_2 \rightarrow w_3), \dots, (w_{k-1} \rightarrow w_k)$ are present in the implication graph, as well as two additional directed edges $(u \rightarrow w_1)$ and $(w_k \rightarrow v)$ for $u, v \in \{w_1, \dots, w_k, \overline{w}_1, \dots, \overline{w}_k\}$.

To show that there are no bicycles, we proceed as in [20], and consider first directed paths of length $\geq L = K\epsilon^{-1} \log n$. Note that a clause with two positive literals adds two directed edges from negative literals to positive literals; a clause with one negative and one positive literal adds one directed edge from a positive to a negative literal and one from a negative to a positive; and a clause with two negative literals adds two directed edge from positive literals to negative literals. So in a path of k literals, if the signs of the literals switch along the path i times there must be $k-1-i$ corresponding clauses with exactly one positive literal, and i corresponding

clauses with two or zero positive literals each, at least $(i-1)/2$ of which must have 0 positive literals. The probability that there is a directed path connecting a given sequence of L literals with i sign changes is therefor

$$\leq q_1^{L-1-i} q_2^{(i+1)/2} q_0^{(i-1)/2}$$

where we have used the fact that $q_0 \leq q_2$. So the expected number of directed paths of length L is bounded above by

$$n^L L \max_i \binom{L}{i} q_1^{L-1-i} q_2^{(i+1)/2} q_0^{(i-1)/2} \quad (2)$$

$$\leq n^2 r^{L-2} L \max_i \binom{L}{i} p_1^{L-1-i} (4p_2 p_0)^{(i-1)/2} \quad (3)$$

Now we consider bicycles of length $\leq L$. The expected number of bicycles of length at most L is bounded above:

$$\mathbb{E}Y \leq \sum_{k=2}^L n^k (q_2 + q_1 + q_0)^2 k^2 \sum_{i=0}^{k-1} \binom{k}{i} q_1^{k-1-i} q_2^{(i+1)/2} q_0^{(i-1)/2} \quad (4)$$

$$\leq \sum_{k=2}^L n^k (q_2 + q_1 + q_0)^2 k^3 \max_{i \leq k-1} \binom{k}{i} q_1^{k-1-i} q_2^{(i+1)/2} q_0^{(i-1)/2} \quad (5)$$

$$\leq \frac{1}{n} \sum_{k=2}^L C k^3 r^{k-1} \max_{i \leq k-1} \binom{k}{i} p_1^{k-1-i} (4p_2 p_0)^{(i-1)/2} \quad (6)$$

Considering (3) and (6), we can make the expected number of paths of length L and bicycles of length at most L both $o(1)$ by choosing the constant in L large enough and by choosing r so that for some $\delta > 0$ and k large:

$$r^k \max_{i \leq k} \binom{k}{i} p_1^{k-i} (4p_2 p_0)^{i/2} < (1 - \delta)^k \quad (7)$$

As a rough bound, it suffices to have

$$r < \frac{1}{2p_1} \text{ and } r < \frac{1}{4\sqrt{p_0}}$$

which translates to

$$r < \frac{2^{kl/2}}{4(k+1)^{(l-1)/2}} \quad (8)$$

With this choice of r , the expected number of bicycles in the implication graph is $o(1)$ and therefor the formula is satisfiable whp.

To prove Theorem 2, we compute the p_i 's and consider (7) more closely. For $k = 3, l = 5$,

$$\begin{aligned} p_2 &= \frac{31}{32} \\ p_1 &= \frac{3}{128} \\ p_0 &= \frac{1}{128} \end{aligned}$$

And thus from (7) we need for k large,

$$\begin{aligned} (1 - \delta)^k &> r^k \max_{i \leq k} \binom{k}{i} \left(\frac{3}{128} \right)^{k-i} \left(\frac{2\sqrt{124}}{128} \right)^i \\ &= r^k \left(\frac{3}{128} \right)^k \max_{i \leq k} \binom{k}{i} \left(\frac{2\sqrt{124}}{3} \right)^i \end{aligned}$$

and so we need

$$r < \frac{128}{3} \min_{\alpha \in [0,1]} \exp \left[-H(\alpha) - \alpha \log(2\sqrt{124}/3) \right]$$

A numerical calculation shows that taking $r = 5.065$ is enough.

For Theorem 3, the p_i 's for the 2-clause, 2-SAT rule are:

$$\begin{aligned} p_2 &= \frac{7}{16} \\ p_1 &= \frac{6}{16} \\ p_0 &= \frac{3}{16} \end{aligned}$$

And thus, similarly to the above, we need

$$r < \frac{8}{3} \min_{\alpha \in [0,1]} \exp [H(\alpha) + \alpha/2 \log(3/7)]$$

which gives $r = 1.055$.

6 Discussion and Open Problems

We conclude with some remarks and open problems.

A first question is whether there is a 2-clause rule to shift the k -SAT threshold for $k > 2$. It is natural to conjecture that in fact the rule used in the proof of Theorem 1 does in fact shift the threshold for $l = 2$, since it does shift the first-moment upper bound. The difficulty lies in the gap between the current upper and lower bounds on $r_k(n)$ - to prove that the rule has shifted

the threshold for 3-SAT, we need to prove that it has shifted the threshold all the way past 4.508. What would be much more straightforward to prove would be that certain algorithms (the unit-clause algorithm and its variants described in [2]) succeed at higher densities for this rule than for random k -SAT.

We have discussed bounds on the satisfiability threshold of Achlioptas processes here and mentioned Friedgut’s result on the sharpness of the k -SAT threshold. In fact the rule we analyze can be shown to have a sharp threshold using Bourgain’s sharp threshold criterion (Bourgain’s appendix to [27]). It would be interesting to determine which Achlioptas k -SAT processes have a sharp threshold or if all rules for a fixed l have a sharp threshold.

Question 2. *For fixed l and k , is there an l -clause Achlioptas rule for k -SAT that does not have a sharp threshold?*

Next we note that the rules we analyze above all operate by biasing the formula to favor a particular assignment.

Question 3. *Can the k -SAT threshold be shifted by an Achlioptas rule that is symmetric with respect to assignments?*

One candidate for such a rule would be the following:

- If all (or none, for the opposite effect) of the literals in the first clause appear in the current formula, add it.
- Otherwise add the second clause.

And finally, analogies to work done on the phase transition of Achlioptas random graphs suggest many avenues for future work.

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